Mathematics 272 Lecture 1 Notes

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1 Teaser of Topics Relating to Combinatorial Limits

The theory of combinatorial limits is a relatively new area. In this course, we will be looking at the following objects:

- graphs
- permutations
- hypergraphs

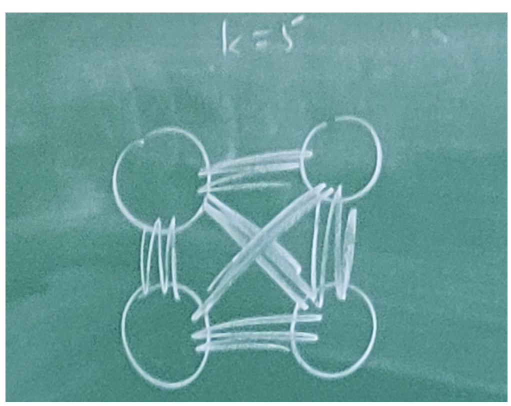
1.1 Edge bounds and Turán's theorem

Let us first start with classical results.

Theorem 1.1 (Mantel, 1907)**.** *Every n-vertex triangle-free graph has $\leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ edges.*

One can actually prove a stronger result.

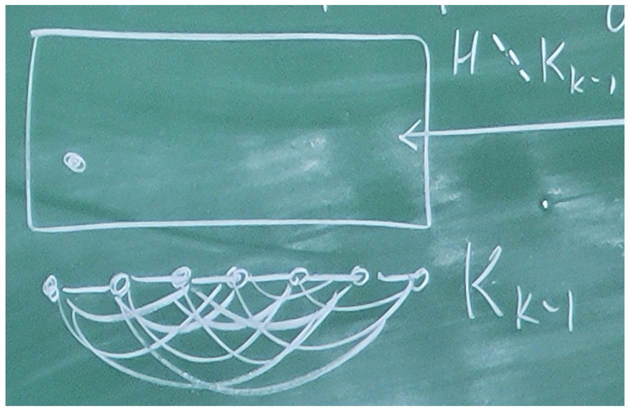
Theorem 1.2 (Turán, 1941)**.** *Every n-vertex graph with no complete graph K_k has $\leq t(n,k)$ edges, where $t(n,k)$ is the number of edges in $(k-1)$-partite complete graph with parts of almost equal sizes.*



These theorems can be interpreted as talking about Turán density.

Definition 1.1. The **Turán density** $\mathrm{ex}(n, H)$ of a graph H is the maximum number of edges of an n-vertex H-free graph.

Proof of Turán's theorem. Fix k, and proceed by induction on n. The graph K_n has $t(n, k)$ edges. For $n \geq k$, consider an n-vertex graph H with no K_k. If H has no K_{k-1}, keep adding edges until K_{k-1} appears.



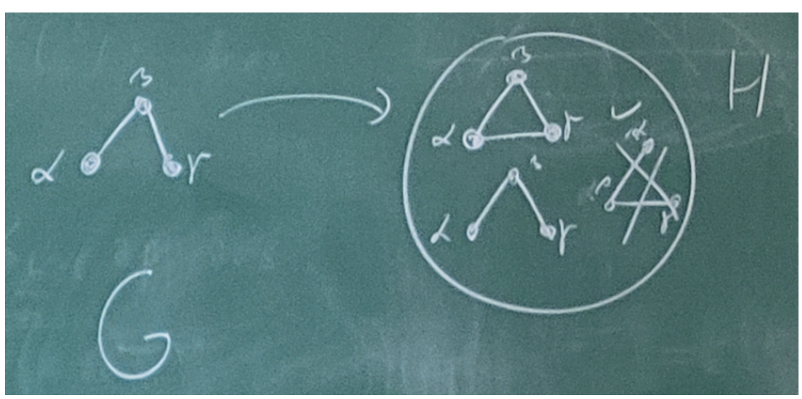
The number of edges in $H \setminus K_{k-1}$ is $t(n-(k-1), k)$, and the remaining edges are at most $(n-(k-1))(k-2) + \binom{k-1}{2}$. Combined, this equals $t(n, k)$. □

Later, we will versions of these theorems that can be proven by a computer.

1.2 p-Quasirandom graphs

This is based on the works of Rődl, Thomason, and Chung–Graham–Wilson.

Definition 1.2. A function $f : V(G) \to V(H)$ is a **graph homomorphism** if for all $u, v \in V(G)$, if $uv \in E(G)$, then $f(u)f(v) \in E(H)$.



Also define $t(G,H)$ to be the probability that a random map $V(G) \to V(H)$ is a homomorphism.

Proposition 1.1. *Let $p \in [0,1]$. The following statements are equivalent for any sequence $(H_n)_{n\in\mathbb{N}}$ of graphs with $|V(H_n)| \to \infty$.*

1. *For every graph G, $\lim_{n\to\infty} t(G,H_n) = p^{|E(G)|}$, which is the expected value of $t(G,H)$ when H is an ErdĦos–Renyi random graph.*

2. *$\lim_{n\to\infty} t(K_2,H_n) = p$ and $\lim_{n\to\infty} t(C_4,H_n) = p^4$, where C_4 is a 4-cycle.*

3. *"Every linear-sized part has density p": For every $\varepsilon, \delta > 0$, there exists an n_0 such that for all $n \geq n_0$ and $W \subseteq V(H_n)$ with $|W| \geq \varepsilon \cdot |V(H_n)|$,*
$$\left| \frac{|E(H_n[W])|}{\binom{|W|}{2}} - p \right| \leq \delta.$$

4. *For every $\varepsilon > 0$, there exists an n_0 such that for all $n \geq n_0$, the eigenvalues $\lambda_1, \ldots, \lambda_{|V(H_n)|}$ of the adjacency matrix A_{H_n} satisfy*
$$\left| \frac{\lambda_1}{|V(H_n)|} - p \right| \leq \varepsilon, \qquad \left| \frac{\lambda_k}{|V(H_n)|} \right| \leq \varepsilon$$
for all $k \geq 2$.

Such a sequence of graphs is called **p-quasirandom**. These statements were originally proven in the 80s without the use of combinatorial limits, but combinatorial limits will allow us to give easier proofs. Moreover, we can use them to prove *asymptotic statements*, such as the following.

Theorem 1.3. *If a large graph G is K_ℓ-free, then $t(K_2,G) \leq \frac{\ell-2}{\ell-1}$.*

Once we introduce the *flag algebra method*, we will see how a computer can prove this (for small enough ℓ; for large ℓ, the computation may take too much time because the search space is too large).

1.3 Regularity decompositions and Roth's theorem

We will need *regularity decompositions* to build (dense) graph and hypergraph limits. Regularity decompositions lead to removal lemmas which enable counting (like with $t(G,H)$). We will be able to prove theorems such as the following.

Theorem 1.4 (Roth, 1953)**.** *For every $\varepsilon > 0$, there exists an n_0 such that for all $n \geq n_0$ and $A \subseteq \{1,\ldots,n\}$, $|A| \geq \varepsilon n$, there exists a length 3 arithmetic progression in A, i.e., $x, y \in A$ such that $x \neq y$ and $\frac{x+y}{2} \in A$.*